On minimal actions of linear fractional and finite simple groups on homology spheres

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Abstract. The paper concerns the problem for which classes of finite groups G, and in particular finite simple groups, the minimal dimension of a faithful action of G on a homology sphere coincides with the minimal dimension of a linear such action on a sphere. For a prime p, we prove that the minimal dimension of a mod p homology sphere with an action of a linear fractional group PSL(2,p) coincides with the minimal dimension of a linear action of PSL(2,p) on a sphere, and obtain a similar result for some types of alternating, symmetric and some other finite simple groups.

1. Introduction

We consider actions of finite groups on homology spheres; all actions will be smooth (or locally linear) and faithful. For a prime p, the linear group SL(2,p) has periodic cohomology, of period the least common multiple of 4 and $p-1$, and hence the minimal possible dimension of an integer homology sphere with a nontrivial *free* SL(2,p)-action is this least common multiple minus one. See [MTW] for the existence of such actions on spheres; since the only finite perfect group which admits a free linear action on a sphere is the binary dodecahedral group SL(2,5), for $p > 5$ such actions are necessarily nonlinear.

In analogy, and motivated also by the determination of the finite groups which admit arbitrary, i.e. not necessarily free actions on low-dimensional homology spheres, in [MZ1] and [Z1] the problem was discussed to determine the minimal dimension of a homology sphere which admits an action of a linear fractional group PSL(2,p); this minimal dimension has been determined in [MZ2] for the groups PSL(2,2^m) where it coincides with the minimal dimension 2^m-2 of a linear action of PSL(2,2^m) on a sphere, and in [Z1] for the groups PSL(2,p) for various small values of the prime p, and for some other finite simple groups. We note that actions of PSL(2,p), and more generally of any finite nonabelian simple group, on homology spheres are necessarily nonfree.

In the following, a mod p homology sphere is a closed n-manifold with the mod p-homology of the n-sphere (i.e. homology with coefficients in the integers $\mathbb{Z}_p$ mod p). Our main result is the following.

Theorem 1. *For an odd prime p, the minimal dimension of an action of* $\mathrm{PSL}(2,p)$ *on a mod p homology sphere coincides with the minimal dimension of a linear action of* $\mathrm{PSL}(2,p)$ *on a sphere (equal to $p-2$ if $p \equiv 3$ mod 4, to $(p-1)/2$ if $p \equiv 1$ mod 4).*

By [DH], Theorem 1 is true for finite p-groups, see also [D] for the case of orientation-preserving actions of finite abelian groups. However, there are finite solvable groups for which the two minimal dimensions do not coincide; specifically, the Milnor groups $Q(8a,b,c)$ ([Mn]) do not admit (faithful, orientation-preserving) linear actions on S^3 (neither free nor nonfree), but by [Mg] some of them admit a free action on a homology 3-sphere. We shall discuss the situation in dimension three in section 4.

The subgroup G of PSL(2,p) represented by all upper triangular matrices is a semidirect product $\mathbb{Z}_p \rtimes \mathbb{Z}_q$, with $q = (p-1)/2$, where $\mathbb{Z}_q$ is the subgroup represented by all diagonal matrices and $\mathbb{Z}_p$ by all matrices with diagonal entries equal to one; we note that $\mathbb{Z}_q$ acts effectively on the normal subgroup $\mathbb{Z}_p$ of G. Now the minimal dimensions of orientation-preserving linear actions on a sphere coincide for the groups $G = \mathbb{Z}_p \rtimes \mathbb{Z}_q$ and PSL(2,p), hence Theorem 1 follows from the following:

Theorem 2. *For an odd prime p and a positive integer q, let $G = \mathbb{Z}_p \rtimes \mathbb{Z}_q$ be a semidirect product with an effective action of $\mathbb{Z}_q$ on the normal subgroup $\mathbb{Z}_p$. The minimal dimension of a faithful action of G on a mod p homology sphere coincides with the minimal dimension of a faithful linear action of G on a sphere (equal to $2q-1$ if q is odd, to q if q is even and G acts orientation-preservingly, to $q-1$ is some element of G reverses the orientation).*

The main tool of the proof of Theorem 2 will be the spectral sequence of the Borel fibration associated to the group action.

The semidirect product $\mathbb{Z}_p \rtimes \mathbb{Z}_q$, with $q = (p-1)/2$ and an effective action of $\mathbb{Z}_q$ on $\mathbb{Z}_p$, is a subgroup also of the alternating group $\mathbb{A}_p$ (considering the action of G on the left cosets of the subgroup $\mathbb{Z}_q$). Similarly, $\mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ is a subgroup of the symmetric group $\mathbb{S}_p$, hence Theorem 2 implies also:

Corollary. *i) For a prime p such that $q = (p-1)/2$ is odd, the minimal dimension of a faithful action of the alternating group $\mathbb{A}_p$ on a homology sphere coincides with the minimal dimension $p-2$ of a faithful linear action of $\mathbb{A}_p$ on a sphere.*
ii) For a prime p, the minimal dimension of a faithful action of the symmetric group $\mathbb{S}_p$ on a homology sphere coincides with the minimal dimension $p-2$ of a faithful linear action of $\mathbb{S}_p$ on a sphere.

In [Z1], this has been shown for the alternating groups $\mathbb{A}_m$ such that m is a power of two or of three. We shall discuss some other finite simple groups in section 3.

We note that the finite simple (and nonsolvable) groups which act on homology spheres in dimension three and four have been determined in [MZ1-3] and [Z2]: the only simple

group which occurs in dimension three is the alternating group $\mathbb{A}_5 \cong$ PSL(2,5), in dimension four there is in addition the group $\mathbb{A}_6 \cong$ PSL(2,9). The classification of the simple groups acting in dimension five is still open. In contrast, it is likely that all groups PSL(2,p) admit an action already on a mod 2 homology 3-sphere, see [Z3,4] for a discussion and various examples for small values of p. We note that every finite group admits a free action on a rational homology 3-sphere, see [CL].

2. The Borel spectral sequence and the proof of Theorem 2

Let G be a finite group acting on a space X. Let EG denote a contractible space on which G acts freely, and $BG = EG/G$. We consider the twisted product $X_G = EG \times_G X = (EG \times X)/G$. The "Borel fibering" $X_G \to BG$, with fiber X, is induced by the projection $EG \times X \to EG$, and the equivariant cohomology of the G-space X is defined as $H^*(X_G; K)$. Our main tool will be the Leray-Serre spectral sequence $E(X)$ associated to the Borel fibration $X_G \to BG$,

$$E_2^{i,j} = E_2^{i,j}(X) = H^i(BG; H^j(X;K)) = H^i(G; H^j(X;K)) \quad \Rightarrow \quad H^{i+j}(X_G;K),$$

i.e. converging to the graded group associated to a filtration of $H^*(X_G;K)$ ("Borel spectral sequence", see e.g. [Br]); here K denotes any abelian coefficient group or commutative ring.

Now suppose that G acts on a (open or closed) n-manifold M; we denote by Σ the singular set of the G-action (all points in M with nontrivial stabilizer). Crucial for the proofs of Theorems 1 and 2 is the following Proposition 1, see [Mc],[E1,2],[E2] for a proof (see also [Bw, Proposition VII.10.1] for a Tate cohomology version).

Proposition 1. *In dimensions greater than n, inclusion induces isomorphisms*

$$H^*(M_G;K) \cong H^*(\Sigma_G;K).$$

We use this to prove the following:

Proposition 2. *For an odd prime p and a positive integer q, let $G = \mathbb{Z}_p \rtimes \mathbb{Z}_q$ be a semidirect product with an effective action of $\mathbb{Z}_q$ on the normal subgroup $\mathbb{Z}_p$. Suppose that G admits a faithful action on a manifold M with the mod p homology of the n-sphere, and that the subgroup $\mathbb{Z}_p$ of G acts freely. Then $n+1$ is a multiple of $2q$ if all elements of G act as the identity on $H^n(M;\mathbb{Z}_p) \cong \mathbb{Z}_p$ (the "orientation-preserving case"), and an odd multiple of q if some element of G acts as minus identity (the "orientation-reversing case").*

Proof. We consider first the Borel spectral sequence $E(\Sigma)$ converging to the cohomology of Σ_G, with $E_2^{i,j} = H^i(G; H^j(\Sigma;\mathbb{Z}_p)$. Let Σ_q denote the singular set of the subgroup $\mathbb{Z}_q$

of G. The singular set Σ of G is the disjoint union of the singular sets of the p conjugates of $\mathbb{Z}_q$ in G; these fixed point sets are all disjoint since the action of $\mathbb{Z}_q$ on $\mathbb{Z}_p$ is effective and the action of $\mathbb{Z}_p$ is free, by assumption. Now the action of G on the cohomology $H^j(\Sigma;\mathbb{Z}_p)$ of the orbit $\mathbb{Z}_p(\Sigma_q)=\Sigma$ is induced (or co-induced) from the action of $\mathbb{Z}_q$ on $H^j(\Sigma_q;\mathbb{Z}_p)$, and by Shapiro's Lemma ([Bw,Proposition III.6.2]), $H^i(G;H^j(\Sigma;\mathbb{Z}_p)$ is isomorphic to $H^i(\mathbb{Z}_q;H^j(\Sigma_q;\mathbb{Z}_p)$ and hence trivial, for $i>0$. So also $H^*(\Sigma_G;\mathbb{Z}_p)$ is trivial, in positive dimensions. By Proposition 1, also $H^*(M_G;\mathbb{Z}_p)\cong H^*(\Sigma_G;\mathbb{Z}_p)$ is trivial, in sufficiently large dimensions.

Next we analyze the spectral sequence $E(M)$ converging to the cohomology of M_G. The E_2-terms $E_2^{i,j}=H^i(G;H^j(M;\mathbb{Z}_p))$ are concentrated in the two rows $j=0$ and $j=n$ where they are equal to $\mathbb{Z}_p$, with a possibly twisted action of G on $H^n(M;\mathbb{Z}_p)\cong\mathbb{Z}_p$. In particular, the only possibly nontrivial differentials of $E(M)$ are $d_{n+1}^{i,j}:E_5^{i,n}\to E_5^{i+n+1,0}$, of bidegree $(n+1,-n)$.

By [Bw, Theorem III.10.3], for $i>0$

$$H^i(G;H^j(M;\mathbb{Z}_p))\cong H^i(\mathbb{Z}_p;H^j(M;\mathbb{Z}_p))^{\mathbb{Z}_q}.$$

The cohomology ring $H^*(\mathbb{Z}_p;\mathbb{Z}_p)$ is the tensor product of a polynomial algebra $\mathbb{Z}_p[t]$ on a 2-dimensional generator t and an exterior algebra $E(s)$ on a 1-dimensional generator s (see [AM, Corollary II.4.2]); also, t is the image of s under the mod p Bockstein homomorphism.

Suppose first that $\mathbb{Z}_q$ acts trivially on $H^n(M;\mathbb{Z}_p)\cong\mathbb{Z}_p$. Since the action of $\mathbb{Z}_q$ on $\mathbb{Z}_p$ is effective, also the action of $\mathbb{Z}_q$ on $H^1(\mathbb{Z}_p;\mathbb{Z}_p)$ and $H^2(\mathbb{Z}_p;\mathbb{Z}_p)$ is effective: denoting by σ the action of a generator of $\mathbb{Z}_q$ on the cohomology $H^i(\mathbb{Z}_p;\mathbb{Z}_p)$, one has $\sigma(s)=ks$ and $\sigma(t)=kt$, for some integer k representing an element of order q in $\mathbb{Z}_p$. Hence $\sigma(t^a)=k^at^a$ and the only powers of t fixed by σ are those divisible by q (in dimensions i which are even multiples of q); similarly, $\sigma(st^a)=k^{a+1}t^{a+1}$, so the only elements in odd dimensions fixed by σ are the products st^a such that $a+1$ is a multiple of q (in dimensions i such that $i+1$ is an even multiple of q). Consequently, $H^i(G;H^j(M;\mathbb{Z}_p))$ is nontrivial exactly for $j=0$ and n and in dimensions i such that either i or $i+1$ is a multiple of $2q$.

Since $H^*(M_G;\mathbb{Z}_p)\cong H^*(\Sigma_G;\mathbb{Z}_p)$ is trivial in sufficiently large dimensions, the differentials d_{n+1} of the spectral sequence $(E_2^{i,j}(M)=H^i(G;H^j(M;\mathbb{Z}_p)$ (concentrated in the rows $j=0$ and n) have to be isomorphisms in large dimensions. This can happen only if $n+1$ is a multiple of $2q$, which completes the proof of Proposition 2 in the orientation-preserving case.

Now suppose that a generator of $\mathbb{Z}_q$ acts as minus identity on $H^n(M;\mathbb{Z}_p)\cong\mathbb{Z}_p$, in particular q is even. Denoting by $\tilde{\sigma}$ the action of a generator of $\mathbb{Z}_q$ on the cohomology

$H^i(\mathbb{Z}_p; H^n(M;\mathbb{Z}_p))$, we now have that $\tilde{\sigma}(t^a) = -\sigma(t^a) = -k^a t^a$, $\tilde{\sigma}(st^a) = -\sigma(st^a) = -k^{a+1}t^{a+1}$. Now the elements of $H^i(\mathbb{Z}_p; H^n(M;\mathbb{Z}_p)) = H^i(\mathbb{Z}_p;\mathbb{Z}_p)$ invariant under $\tilde{\sigma}$ are the powers of t by odd multiples of $q/2$ (in dimensions i which are odd multiples of q), and the products st^a such that $a+1$ is an odd multiple of $q/2$ (in dimensions i such that $i+1$ is an odd multiple of q). Hence $H^i(G; H^j(M;\mathbb{Z}_p))$ is nontrivial exactly in the following situations: either $j=0$ and i or $i+1$ is an even multiple of q (since the action of G on $H^0(M;\mathbb{Z}_p)$ is trivial), or if $j=n$ and i or $i+1$ is an odd multiple of q (with the twisted action of G on $H^n(M;\mathbb{Z}_p)$). Since again $H^*(M_G;\mathbb{Z}_p) \cong H^*(\Sigma_G;\mathbb{Z}_p)$ is trivial in large dimensions, the differential d_{n+1} has to be an isomorphism and $n+1$ an odd multiple of q. (Note that, in order to obtain just the lower bound $n \geq q-1$, one may apply the orientation-preserving case to the subgroup $\mathbb{Z}_p \rtimes \mathbb{Z}_{q/2}$ of $\mathbb{Z}_p \rtimes \mathbb{Z}_q$.)

This completes the proof of Proposition 2.

Proof of Theorem 2.

Suppose that $G = \mathbb{Z}_p \rtimes \mathbb{Z}_q$ acts faithfully on a mod p homology n-sphere M. If the subgroup $\mathbb{Z}_p$ of G acts freely, Proposition 2 implies that $n+1$ is a multiple of $2q$ if G acts orientation-preservingly, and an odd multiple of q otherwise; in particular, the minimal possibilities are $n = 2q-1$ and $n = q-1$, respectively.

Suppose then that $\mathbb{Z}_p$ has nonempty fixed point set F. By Smith fixed point theory, F is a mod p homology sphere of some dimension d, $0 \leq d < n$; by duality, the complement $M - F$ is a G-invariant manifold with the mod p homology of a sphere of dimension $n-d-1$. Proposition 2 implies now that $n-d-1 \geq 2q-1$ if q is odd, and that $n-d-1 \geq q-1$ if q is even.

This completes the proof of Theorem 2 (and also of Theorem 1 and the Corollary).

3. Some other finite simple groups

We collect some other finite simple or closely related groups for which the two minimal dimensions coincide.

Proposition 3. *The minimal dimension* $\mathrm{hd}(G)$ *of a faithful action of* G *on a (integer) homology sphere coincides with the minimal dimension* $\mathrm{ld}(G)$ *of a faithful linear action of* G *on a sphere for the following finite groups:*
the Mathieu groups M_{11} *and* M_{23}*;*
the unitary and symplectic groups $\mathrm{PSU}(3,3)$, $\mathrm{PSU}(4,2) \cong \mathrm{PSp}(4,3)$ *and* $\mathrm{PSp}(6,2)$*;*
the Weyl or Coxeter groups E_6, E_7 *and* E_8 *of the corresponding exceptional Lie algebras.*

For the two Matthieu groups this follows from Theorem 2, noting that M_{11} has subgroups $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5 \subset \mathrm{PSL}_2(11)$ and a linear action on S^9, M_{23} a subgroup $\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11}$ and a linear action on S^{21}. For most of the other groups, it is a consequence of the following well-known result from Smith fixed point theory (see [Sm]).

Proposition 4. *i) For an odd prime number p, the minimal dimension of a faithful action of the group $(\mathbb{Z}_p)^k$ on a mod p homology sphere is $2k-1$.*
ii) The minimal dimension of a faithful action of $(\mathbb{Z}_2)^k$ on a mod 2 homology sphere is $k-1$.
iii) The minimal dimension of a faithful, orientation-preserving action of $(\mathbb{Z}_2)^k$ on a mod 2 homology sphere is k.

We remark that, though we consider smooth actions in the present paper, at least parts i) and ii) remain true for arbitrary continuous actions (see also the comments in [BV, section 4.2]).

We refer to [C] for information about the subgroup structure and the character tables of the finite simple groups. The unitary group $\mathrm{PSU}(4,2)$ has a maximal subgroup $(\mathbb{Z}_3)^3$ and a linear action on a sphere of dimension 5, so Proposition 4 implies that $\mathrm{hd}(G) = \mathrm{ld}(G) = 5$; also, $\mathrm{PSU}(4,2)$ is a subgroup of index two in the Weyl group of type E_6 which has also a linear action on a 5-dimensional sphere, so again $\mathrm{hd}(G) = \mathrm{ld}(G) = 5$.

Similarly, the Weyl group of type E_8 has a subgroup $(\mathbb{Z}_3)^4$ and a linear action on a sphere of dimension 7 (see the closely related orthogonal group $\mathrm{O}_8^+(2)$ in [C]), so $\mathrm{hd}(G) = \mathrm{ld}(G) = 7$ in this case.

The symplectic group $\mathrm{PSp}(6,2)$ is a subgroup of index 2 in the Weyl group of type E_7, and both act linearly on a sphere of dimension 6. Now these groups have as subgroups the alternating group $\mathbb{A}_8$ and the linear group $\mathrm{PSL}(2,8)$; by [MZ2, Prop.1] and [Z1, Corollary 2], the minimal dimension $\mathrm{hd}(G)$ is 6 for both $\mathbb{A}_8$ and $\mathrm{PSL}(2,8)$, so $\mathrm{hd}(G) = \mathrm{ld}(G) = 6$ for all these groups. Alternatively, $\mathrm{PSp}(6,2)$ has a subgroup $(\mathbb{Z}_2)^6$, so one may apply Proposition 4 iii).

Finally, we consider the unitary group $\mathrm{PSU}(3,3)$ which admits a linear action on a sphere of dimension 6. We consider a subgroup $A \cong (\mathbb{Z}_3)^2$ of $\mathrm{PSU}(3,3)$. By Proposition 4, A and hence $\mathrm{PSU}(3,3)$ do not act faithfully on a homology n-sphere if $n < 5$. Suppose, by contradiction, that $\mathrm{PSU}(3,3)$ acts on a homology 5-sphere M. We apply the Borel formula to the subgroup A of $\mathrm{PSU}(3,3)$ ([Bo, Theorem XIII.2.3]). For every elementary abelian p-group A acting on a mod p homology n-sphere, the Borel formula states that

$$n - r = \Sigma_H (r(H) - r)$$

where the sum is taken over all subgroups H of index p of A, $r(H)$ denotes the dimension of the fixed point set of H and r that of A (equal to -1 if the fixed point set is empty), so in our situation $n = 5$ and $H \cong \mathbb{Z}_3$. We note that $\mathrm{PSU}(3,3)$ has two conjugacy classes of elements of order 3, and that $r(H) = 3, 1$ or -1 (since H is orientation-preserving, its fixed point set has even codimension). Then one checks easily that the only solution of the Borel equation is for $r = 1$ and the values 3, 3, 1, 1 for the four subgroups $H \cong \mathbb{Z}_3$ of A. Hence the fixed point set of A is a 1-sphere S^1 in M; taking an invariant tubular

neighbourhood, A acts on a 4-disk transverse to S^1 in some (any) point of S^1 and on its boundary, a 3-sphere S^3. By Proposition 4, $A \cong (\mathbb{Z}_3)^2$ does not act faithfully on a 3-sphere, so some element of A acts trivially on the transversal 4-disk of the fixed point set S^1 of A, and hence trivially on the whole tubular neighbourhood of S^1 and then also on M which is a contradiction. We have shown that $\mathrm{PSU}(3,3)$ does not act on a homology 5-sphere, so the minimal dimension $\mathrm{hd}(G)$ is equal to 6.

This completes the proof of Proposition 3.

4. Some comments on the situation in dimension three

If a finite group G acts freely on a homology 3-sphere then G has periodic cohomology of period four and a unique involution; a list of such groups is given in [Mn], as well as a list of all groups which admit a faithful linear action on the 3-sphere.

We consider the class of groups $Q(8a,b,c)$ in [Mn] which have periodic cohomology of period four but do not admit a faithful linear action on S^3. The group $Q(8a,b,c)$ has a presentation

$$< x, y, z \,|\, x^2 = (xy)^2 = y^{2a}, z^{bc} = 1, xzx^{-1} = z^r, yzy^{-1} = z^{-1} >$$

for relatively coprime positive integers $8a, b$ and c such that either a is odd and $a > b > c \geq 1$, or $a \geq 2$ is even and $b > c \geq 1$; also, $r \equiv -1 \mod b$ and $r \equiv +1 \mod c$. Note that $Q(8a,b,c)$ is a semidirect product with normal subgroup $\mathbb{Z}_b \times \mathbb{Z}_c \cong \mathbb{Z}_{bc}$ (generated by z) and factor group the generalized quaternion or binary dihedral subgroup $Q(8a) \cong Q(8a,1,1)$ of order $8a$ (generated by x and y.

It is shown in [L] that, if a is even, then $Q(8a,b,c)$ does not admit a free action on a homology 3-sphere. If a is odd then $Q(8a,b,c)$ is an extension of $\mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c \cong \mathbb{Z}_{abc}$ by the quaternion group $Q(8) = \{\pm 1, \pm i, \pm j, \pm k\}$, and i, j and k acts trivially on $\mathbb{Z}_a, \mathbb{Z}_b$ and $\mathbb{Z}_c$, respectively, and in a dihedral way on the other two.

Lemma 1. *A group $Q(8a,b,c)$ does not admit a faithful orientation-preserving nonfree action on a homology 3-sphere.*

Proof. Suppose that $G = Q(8a,b,c)$ acts orientation-preservingly on a homology 3-sphere M. The unique involution $h = x^2 = (xy)^2 = y^{2a}$ of G is central in G; by Smith fixed point theory the fixed point set of h is either empty or a simple closed curve K in M (see e.g. [Br]). Suppose that h has nonempty connected fixed point set K; note that K is invariant under the action of G. Now if a finite orientation-preserving group leaves invariant a simple closed curve K in a 3-manifold then G is isomorphic to a subgroup of a semidirect product $\mathbb{Z}_2 \ltimes (\mathbb{Z}_m \times \mathbb{Z}_n)$, with a dihedral action of $\mathbb{Z}_2$ on $\mathbb{Z}_m \times \mathbb{Z}_n$ ($\mathbb{Z}_2$ acts as a reflection or strong inversion on K whereas $\mathbb{Z}_m \times \mathbb{Z}_n$ acts by rotations around

and along K). Clearly $G = Q(8a, b, c)$ is not of this type, hence the unique involution h of G has to act freely on M and then also every nontrivial element of even order in G.

Next suppose that some nontrivial element g in G of odd prime order has nonempty connected fixed point set K; up to conjugation, we can assume that g is an element of one of the subgroups $\mathbb{Z}_b \times \mathbb{Z}_c$ or $Q(8a) \cong Q(8a, 1, 1)$ of G. In both cases, some element i (or j or k) of order four in the generalized quaternion group $Q(8a)$ acts dihedrally on g (that is, $igi^{-1} = g^{-1}$), and hence acts as a reflection or strong inversion on the fixed point set K of g. But then also i has nonempty fixed point set which is a contradiction.

This finishes the proof of Lemma 1.

By the recent geometrization of the free finite group actions on the 3-sphere due to Perelman (or equivalently, by the geometrization of the closed 3-manifolds with finite fundamental group), none of the groups $Q(8a, b, c)$ admits a free action on S^3 (since they admit no free linear actions). However, it is implied by the results in [Mg] that some of the groups $Q(8a, b, c)$ admit a free action on a homology 3-sphere (see also the comments in [K, Problem 3.37 Update A (p.173)]). Since, as a consequence of Lemma 1, none of the groups $Q(8a, b, c)$ admits a linear nonfree action on the 3-sphere, it follows then from the results in [Mg] that for some of the groups $Q(8a, b, c)$ the minimal dimension $\mathrm{hd}_+(G)$ of a faithful, orientation-preserving action on an integer homology sphere is strictly smaller than the minimal dimension $\mathrm{ld}_+(G)$ of a faithful, orientation-preserving linear action on a sphere.

What appears challenging is the fact that, in these examples of type $Q(8a, b, c)$, the minimal dimension of an action on a homology 3-sphere is realized only by a free action: what about groups which admit only nonfree actions on homology spheres, or do not have periodic cohomology? We do not know any example of such a group for which $\mathrm{hd}_+(G)$ is strictly smaller than $\mathrm{ld}_+(G)$. For example, does the central product (the direct product with identified central involutions)

$$Q(8a, b, c) \times_{\mathbb{Z}_2} \mathbb{Z}_4$$

act on a homology 3-sphere? (assuming that $Q(8a, b, c)$ does.) This group does not act freely on a homology 3-sphere (because it has a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$), and it does not act linearly on S^3 (since $Q(8a, b, c)$ does not).

We discuss now a related problem on finite group actions on homology 3-spheres and other possible examples. It is shown in [Z2] that the finite nonsolvable groups which admit actions on homology 3-spheres are exactly the finite nonsolvable subgroups of the orthogonal group $\mathrm{SO}(4) \cong S^3 \times_{\mathbb{Z}_2} S^3$ (the central product of two copies of the unit quaternions), plus possibly two other classes of groups which are the central products

$$\mathbb{A}_5^* \times_{\mathbb{Z}_2} Q(8a, b, c)$$

where a is odd ($\mathbb{A}_5^*$ denotes the binary dodecahedral group), and their subgroups

$$\mathbb{A}_5^* \times_{\mathbb{Z}_2} (\mathbb{D}_{4a}^* \times \mathbb{Z}_b)$$

($\mathbb{D}_{4a}^*$ denotes the binary dihedral or generalized quaternion group of order $4a$). In turn these have a subgroup

$$\mathbb{D}_8^* \times_{\mathbb{Z}_2} (\mathbb{D}_{4a}^* \times \mathbb{Z}_b)$$

which does not act freely on a homology 3-sphere (since it has a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$); also, the following holds:

Lemma 2. *For odd coprime integers $a, b \geq 3$, the group $G = \mathbb{D}_8^* \times_{\mathbb{Z}_2} \mathbb{D}_{4a}^* \times \mathbb{Z}_b$ does not admit a faithful orientation-preserving linear action on S^3.*

Proof. Suppose that G is a subgroup of the orthogonal group $\mathrm{SO}(4) \cong S^3 \times_{\mathbb{Z}_2} S^3$. The finite subgroups of the unit quaternions S^3 are cyclic, binary dihedral or binary polyhedral groups. The two "projections" of the subgroup $\mathbb{D}_8^*$ of G to the first and second factor of $S^3 \times_{\mathbb{Z}_2} S^3$ are cyclic or binary dihedral groups; since $\mathbb{D}_8^*$ is nonabelian, one of the two projections, say the first one, has to be a binary dihedral group. Now, since the projections of the subgroups $\mathbb{D}_8^*$ and $\mathbb{D}_{4a}^*$ of G commute elementwise (modulo the central involution), the projection of $\mathbb{D}_{4a}^*$ to the second factor of $S^3 \times_{\mathbb{Z}_2} S^3$ has to be a binary dihedral group (and the projection to the first factor has to be trivial). But then at least one of the two projections of the cyclic subgroup $\mathbb{Z}_b$ of G (any nontrivial one) does not commute elementwise with either the binary dihedral projection of $\mathbb{D}_8^*$ or that of $\mathbb{D}_{4a}^*$. This contradiction completes the proof Lemma 2.

Question. Suppose that a, b are greater than one, odd and coprime. Does

$$\mathbb{D}_8^* \times_{\mathbb{Z}_2} \mathbb{D}_{4a}^* \times \mathbb{Z}_b$$

admit a faithful, orientation-preserving action on a homology 3-sphere? (if a is even then there is no such action by [Z2, Lemma].)

If the answer is no then, by [Z2], the class of the nonsolvable groups which admit an action on a homology 3-sphere coincides exactly with the class of the nonsolvable subgroups of the orthogonal $\mathrm{SO}(4)$. On the other hand, if the group admits such an action then $\mathrm{hd}_+(G)$ is strictly smaller than $\mathrm{ld}_+(G)$ for a group G which admits only nonfree actions on a homology sphere.

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